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Monomial Hopf algebras over fields of positive characteristic

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Abstract In this paper, we study the structures of monomial Hopf algebras over a field of positive characteristic. A necessary and sufficient condition for the monomial coalgebra $C_d(n)$ to admit Hopf structures is given here, and if it is the case, all graded Hopf structures on $C_d(n)$ are completely classified. Moreover, we construct a Hopf algebras filtration on $C_d(n)$ which will help us to discuss a conjecture posed by Andruskiewitsch and Schneider. Finally combined with a theorem by Montgomery, we give the structure theorem for all monomial Hopf algebras.

Keywords: monomial coalgebra, Gaussian binomial coefficient, monomial Hopf algebra.

1 Introduction

There are several works to construct neither commutative nor cocommutative Hopf algebras via quivers (see refs. [1–3]). An advantage for this construction is that a natural basis consisting of paths is available, and one can relate the properties of a quiver to the ones of the corresponding Hopf structures.

In ref. [1], the authors have classified all the finite-dimensional Hopf structures on a monomial algebra, or equivalently, on a monomial coalgebra over a field of characteristic zero and containing all roots of unity. As a continuation of ref. [1], we want to classify Hopf structures on a monomial coalgebra when the characteristic of the field is positive.

On the one hand, we note that there do exist Hopf structures on a monomial coalgebra when the characteristic of the field is not zero (see Example 1). On the other hand, we note that there exists an essential difference on the monomial Hopf structures when the characteristic of the base field is different. For example, we can get examples of finite-dimensional monomial (pointed) Hopf algebras which cannot be generated by group-like and primitive elements when the characteristic of the base field is p while in the characteristic zero case, such examples do not exist any more (see sec. 3).

Just like that in ref. [1], our main task is to study the Hopf structures on $C_d(n)$, where $C_d(n)$ is the sub-coalgebra of the path coalgebra kZ_n^c with the set of all paths of length strictly smaller than d as a basis (see sec. 2). It turns out that the coalgebra $C_d(n)$ admits a Hopf structure if and only if there exist a d_0-th primitive root of unity $q \in k$ with $d_0|n$ and a natural number $r \geqslant 0$ such that $d = p^r d_0$, where p is the characteristic of the base field k (Theorem 1). Consequently, we obtain a Hopf algebras filtration for $C_d(n)$, which will help us to discuss a conjecture raised by Andruskiewitsch and Schneider. We give all the graded (with length grading) Hopf structures on $C_d(n)$ (see Theorem 2). As for the non-graded case, we cannot give them all. But we show that there do exist non-graded structures on $C_d(n)$ (see Example 2).

Next we study the monomial Hopf algebras, and show that for a given monomial Hopf algebra, each indecomposable component as coalgebras is isomorphic to $C_d(n)$ for some n, d with $d \geqslant 2$ or the field k simultaneously (see Lemma 6). Finally, by a theorem of Montgomery (see Theorem 3.2 in ref. [4]), we can describe the structures of monomial Hopf algebras.

2 Preliminaries

Throughout this paper, k denotes a field of characteristic p. By an algebra we always mean a finite-dimensional associative k-algebra with an identity element.

First we recall some basic facts, here we follow the definitions and notations in ref. [1].

Quivers considered here are always finite. Given a quiver $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 the set of arrows, denote by kQ, kQ^a and kQ^c the k-space with the set of all paths as a basis in Q, the path algebra of Q, and the path coalgebra of Q, respectively. Note that they are all graded with respect to the length grading. For $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote respectively the starting and the terminating vertices of α.

Recall that the comultiplication of the path coalgebra kQ^c is defined by

$$\Delta(p) = \sum_{\beta\alpha=p} \beta \otimes \alpha = \alpha_l \cdots \alpha_1 \otimes s(\alpha_1) + \sum_{i=1}^{l-1} \alpha_l \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 + t(\alpha_l) \otimes \alpha_l \cdots \alpha_1$$

for any path $p = \alpha_l \cdots \alpha_1$ with each $\alpha_i \in Q_1$, for $i \in \{1, \cdots, l\}$; and $\varepsilon(p) = 0$ for $l \geqslant 1$ and 1 if $l = 0$ ($l = 0$ means that p is a vertex). This is a pointed coalgebra, i.e. a coalgebra over which all simple comodules are of one dimensional.

Let C be a coalgebra. The set of group-like elements is defined to be

$$G(C) := \{c \in C | \Delta(c) = c \otimes c,\ c \neq 0\}.$$

Obviously $\varepsilon(c) = 1$ for $c \in G(C)$. For $x, y \in G(C)$, denote by

$$P_{x,y}(C) := \{c \in C | \Delta(c) = c \otimes x + y \otimes c\}$$

the set of x, y-primitive elements in C. It is clear that $\varepsilon(c) = 0$ for $c \in P_{x,y}(C)$. Note that $k(x-y) \subseteq P_{x,y}(C)$. An element $c \in P_{x,y}(C)$ is non-trivial if $c \notin k(x-y)$. Note that $G(kQ^c) = Q_0$, and

Lemma 1. See Lemma 1.1 in ref. [1]. Let Q be a quiver. For $x, y \in Q_0$, we have $P_{x,y}(kQ^c) = y(kQ_1)x \oplus k(x-y)$, where $y(kQ_1)x$ denotes the k-space spanned by all arrows from x to y. In particular, there is a non-trivial x, y-primitive element in kQ^c if and only if there is an arrow from x to y in Q.

An ideal I of kQ^a is admissible if $J^N \subseteq I \subseteq J^2$ for some positive integer $N \geqslant 2$, where J is the ideal generated by all arrows. An algebra A is elementary if $A/R \cong k^n$ as algebras for some n, where R is the Jacobson radical of A. For an elementary algebra A, there is a (unique) quiver Q and an admissible ideal I of kQ^a, such that $A \cong kQ^a/I$ (see ref. [5]).

An algebra A is monomial if there exists an admissible ideal I generated by some paths in Q such that $A \cong kQ^a/I$. Dually, the authors of ref. [1] gave the definition of monomial coalgebras.

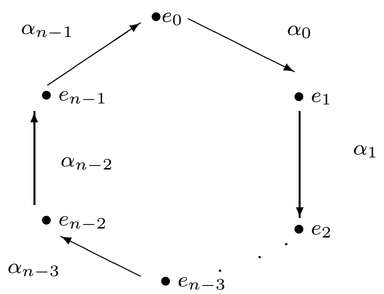
Definition 1. A subcoalgebra C of kQ^c is called monomial provided that the following conditions are satisfied:

(1) C contains all vertices and arrows in Q;

(2) C is contained in the subcoalgebra $C_d(Q) := \oplus_{i=0}^{d-1} kQ(i)$ for some $d \geqslant 2$, where $kQ(i)$ is the k-space spanned by all paths of length i in Q;

(3) C has a basis consisting of paths.

Consider the following quiver



We denote this quiver by Z_n and call it the basic cycle of length n. Denote by p_i^l the path in Z_n of length l starting at e_i. Thus we have $p_i^0 = e_i$ and $p_i^1 = \alpha_i$.

For each n-th root of unity $q \in k$, Cibils and Rosso[2] have defined a graded Hopf algebra structure $kZ_n(q)$ (with the length grading) on the path coalgebra kZ_n^c by

$$p_i^l \cdot p_j^m = q^{jl} \binom{m+l}{l}_q p_{i+j}^{l+m},$$

and the antipode S mapping p_i^l to $(-1)^l q^{-\frac{l(l+1)}{2} - il} p_{n-l-i}^l$, where $\binom{m+l}{l}_q$ is the q-analogue Gaussian binomial coefficient. Recall the definition $\binom{m+l}{l}_q := \frac{(l+m)!_q}{l!_q m!_q}$, where

$l!_q = 1_q \cdot 2_q \cdots l_q$ and $l_q = \frac{1-q^l}{1-q} = 1 + q + \cdots + q^{l-1}$.

Then, we always denote $C_d(Z_n)$ by $C_d(n)$. That is, $C_d(n)$ is the subcoalgebra of kZ_n^c with the set of all paths of length strictly less than d as the basis.

Clearly, if $\binom{m+l}{l}_q = 0$ for all $0 < m, l < d$ and $m+l \geqslant d$, then $C_d(n)$ will be a graded sub-Hopf algebra of $kZ_n(q)$.

Example 1. Let q be a d_0-th primitive root of unity with $d_0|n$. Assume $q \in k$. In the next section (Proposition 1), we will prove that if $d = p^t d_0$ for some nonnegative integer t, then $\binom{d}{l}_q = 0$ for all $0 < l < d$. By a standard identity about the Gaussian binomial coefficients (see ref. [6]), say

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q,$$

we have $\binom{m+l}{l}_q = 0$ for all $0 < m, l < d$ and $m + l \geqslant d$. Therefore, by the discussion above, $C_{p^t d_0}(n)$ forms a graded sub-Hopf algebra of $kZ_n(q)$. We denote this Hopf algebra by $C(d_0, t, n, q)$ or $C(p^t d_0, n, q)$.

The following fact (see Lemma 2.3 in ref. [1]) shows the importance of $C_d(n)$.

Lemma 2. Let C be an indecomposable monomial coalgebra. Then C is coFrobenius (i.e. C^* is Frobenius) if and only if $C = k$ or $C \cong C_d(n)$ for some positive integers n and d with $d \geqslant 2$.

The following lemma (Lemma 3.3 in ref. [1]) is needed in our proof of Theorem 1.

Lemma 3. Suppose that there is a Hopf algebra structure on $C_d(n)$. Then up to a Hopf algebra isomorphism, we have

$$p_i^l \cdot p_j^m \equiv q^{jl} \binom{m+l}{l}_q p_{i+j}^{l+m} \pmod{C_{l+m}(n)}$$

for $0 \leqslant i, j \leqslant n-1$, and for $l, m \leqslant d-1$, where $q \in k$ is an n-th root of unity.

3 Hopf structures on $C_d(n)$ and Andruskiewitsch-Schneider conjecture

The aim of this section is to give an equivalent condition for $C_d(n)$ to admit a Hopf structure (Theorem 1), and then classify all the graded Hopf structures on $C_d(n)$. Moreover, we will construct a Hopf algebras filtration of $C_d(n)$ which will help us to discuss a conjecture posed by Andruskiewitsch and Schneider.

For a positive rational number x, we denote by $[x]$ the largest natural number which is not greater than x.

Let q be a d_0-th primitive roots of unity. Note that $\binom{m+l}{l}_q$ is a polynomial of q. Thus $\binom{m+l}{l}_q = 0$ if and only if the multiplicity of q as a zero point is at least 1; if and only if the multiplicity of q as a zero point of $(l+m)!_q$ is strictly greater than the one of $l!_q m!_q$. Clearly, for any natural number n, $n_q = 0$ if and only if $d_0|n$, and if $n = p^i d_0 d$, where p is coprime to $d_0 d$, then $n_q = \frac{(1-q^{p^i d_0 d})}{(1-q)} = \frac{(1-q^{d_0 d})^{p^i}}{(1-q)}$, thus the multiplicity of q as a zero point of n_q is p^i. By the direct computation, one may show that the multiplicity of q as a zero point in $m!_q$ is given by

$$\left[\frac{m}{d_0}\right]+(p-1)\left[\frac{m}{pd_0}\right]+(p^2-p)\left[\frac{m}{p^2d_0}\right]+\cdots+(p^i-p^{i-1})\left[\frac{m}{p^id_0}\right]+\cdots,$$

and hence we get the following lemma easily.

Lemma 4. Let q be a d_0-th primitive root of unity. Then $\binom{m+l}{l}_q=0$ if and only if

$$\left[\frac{m+l}{p^id_0}\right]>\left[\frac{m}{p^id_0}\right]+\left[\frac{l}{p^id_0}\right]$$

for some $i\geqslant 0$; if and only if

$$\begin{aligned}&\left[\frac{m+l}{d_0}\right]+\left[\frac{m+l}{pd_0}\right]+\left[\frac{m+l}{p^2d_0}\right]+\cdots+\left[\frac{m+l}{p^id_0}\right]+\cdots\\&-(\left[\frac{m}{d_0}\right]+\left[\frac{m}{pd_0}\right]+\left[\frac{m}{p^2d_0}\right]+\cdots+\left[\frac{m}{p^id_0}\right]+\cdots)\\&-(\left[\frac{l}{d_0}\right]+\left[\frac{l}{pd_0}\right]+\left[\frac{l}{p^2d_0}\right]+\cdots+\left[\frac{l}{p^id_0}\right]+\cdots)>0.\end{aligned}$$

Lemma 5. Let $m>1$ be a positive integer. For any $0<n<m$, set

$$\begin{aligned}I_{m,n}=&[m]+\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\cdots+\left[\frac{m}{p^i}\right]+\cdots\\&-([n]+\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^i}\right]+\cdots)\\&-([m-n]+\left[\frac{m-n}{p}\right]+\left[\frac{m-n}{p^2}\right]+\cdots+\left[\frac{m-n}{p^i}\right]+\cdots).\end{aligned}$$

Then $I_{m,n}>0$ for all $0<n<m$ if and only if $m=p^t$ for some $t\geqslant 1$.

Proof. First note that $\left[\frac{m}{p^i}\right]-\left[\frac{n}{p^i}\right]-\left[\frac{m-n}{p^i}\right]\geqslant 0$ for all $i\in N$.

"If Part:" To prove the conclusion, it is enough to find some $j\in N$ such that $\left[\frac{m}{p^j}\right]-\left[\frac{n}{p^j}\right]-\left[\frac{m-n}{p^j}\right]>0$. In fact, let $j=t$, $1=\left[\frac{p^t}{p^t}\right]>\left[\frac{n}{p^t}\right]+\left[\frac{m-n}{p^t}\right]=0$ for all $0<n<m$. Thus, we prove the sufficiency.

"Only if Part:" Clearly, $p\leqslant m$. At first, we claim that $p|m$. Otherwise, assume $m=kp+r$ with $k\geqslant 1$ and $0<r<p$. Let $n=kp$, then it is easy to see that $I_{m,n}=0$ now. It is contradict to the assumption.

Thus, generally, let $m=p^r(a_lp^l+\cdots+a_1p+a_0)$, where $r\geqslant 1$ and $a_i<p$ for $i=0,1,\ldots,l$. Let $n=a_0p^r$ and then $m-n=p^r(a_lp^l+\cdots+a_1p)$. Then, for any $0\leqslant j\leqslant r$, $\left[\frac{m}{p^j}\right]=p^{r-j}(a_lp^l+\cdots+a_1p+a_0)$, $\left[\frac{m-n}{p^j}\right]=p^{r-j}(a_lp^l+\cdots+a_1p)$ and $\left[\frac{n}{p^j}\right]=a_0p^{r-j}$. This implies $\left[\frac{m}{p^j}\right]=\left[\frac{m-n}{p^j}\right]+\left[\frac{n}{p^j}\right]$ when $j\leqslant r$. If $j>r$, then $m=p^j(a_lp^{l-(j-r)}+\cdots+a_{j-r})+a_{j-r-1}p^{j-1}+\cdots+a_0p^r$. But, $a_{j-r-1}p^{j-1}+\cdots+a_0p^r\leqslant(p-1)p^{j-1}+\cdots+(p-1)p^r=p^j-p^r<p^j$. Thus $\left[\frac{m}{p^j}\right]=a_lp^{l-(j-r)}+\cdots+a_{j-r}$, $\left[\frac{m-n}{p^j}\right]=a_lp^{l-(j-r)}+\cdots+a_{j-r}$ and $\left[\frac{n}{p^j}\right]=0$. This implies $\left[\frac{m}{p^j}\right]=\left[\frac{m-n}{p^j}\right]+\left[\frac{n}{p^j}\right]$ for

$j > r$. Summarizing the above discussion, we have $\left[\frac{m}{p^j}\right] = \left[\frac{n}{p^j}\right] + \left[\frac{m-n}{p^j}\right]$ for all j and thus $I_{m,n} = 0$. It is contradict to the assumption. Therefore we know that $a_0 = 0$ or $a_i = 0$ for all $l \leqslant i \leqslant 1$. We claim there is only one $a_i \neq 0$. In fact, if $a_0 \neq 0$, then the conclusion above asserts $a_i = 0$ for all $l \leqslant i \leqslant 1$. If $a_0 = 0$, then repeating the above discussion shows that $a_1 = 0$ or $a_i = 0$ for all $l \leqslant i \leqslant 2$. So, at last, we have a unique a_i such that $m = a_i p^{r+i}$.

If $a_i > 1$, then we can write $a_i = l_1 + l_2$ with $l_1 l_2 \neq 0$. Let $n = l_1 p^{r+i}$, then it is easy to see that $I_{m,n} = 0$. It is also contradict to the assumption. Thus $m = p^{r+i}$ and we get the desired conclusion. □

With these preparations, we can give the following key proposition to our main result (Theorem 1) of this section.

Proposition 1. Let $q \in k$ be a d_0-th primitive root of unity. Then $\binom{d}{n}_q = 0$ for all $0 < n < d$ if and only if $d = p^r d_0$ for some nonnegative integer r.

Proof. For simplicity, denote

$$\left[\frac{m}{d_0}\right] + \left[\frac{m}{pd_0}\right] + \left[\frac{m}{p^2 d_0}\right] + \cdots + \left[\frac{m}{p^i d_0}\right] + \cdots$$

$$-(\left[\frac{n}{d_0}\right] + \left[\frac{n}{pd_0}\right] + \left[\frac{n}{p^2 d_0}\right] + \cdots + \left[\frac{n}{p^i d_0}\right] + \cdots)$$

$$-(\left[\frac{m-n}{d_0}\right] + \left[\frac{m-n}{pd_0}\right] + \left[\frac{m-n}{p^2 d_0}\right] + \cdots + \left[\frac{m-n}{p^i d_0}\right] + \cdots)$$

by $I_{m,n,q}$.

"If Part:" Similarly to the proof of Lemma 5, $1 = \left[\frac{p^r d_0}{p^r d_0}\right] > \left[\frac{n}{p^r d_0}\right] + \left[\frac{p^r d_0 - n}{p^r d_0}\right] = 0$ for all $0 < n < p^r d_0$. That is to say, $I_{d,n,q} > 0$ for all $n < d$ and thus $\binom{d}{n}_q = 0$ for all $0 < n < d$ according to Lemma 4.

"Only if Part:" Clearly, $d \geqslant d_0$. We claim $d_0 | d$. If not, then $d = kd_0 + r$ with $k \geqslant 1$ and $0 < r < d_0$. Let $n = kd_0$, then it is easy to see that $I_{d,n,q} = 0$ and thus $\binom{d}{n}_q \neq 0$ by Lemma 4. It is contradict to the assumption.

Now we have $\frac{d}{d_0}$ is a positive integer and denote it by m. If $m = 1$, then $d = p^0 d_0$. If $m > 1$, then Lemma 4 and Lemma 5 assert that $m = p^r$ and thus $d = md_0 = p^r d_0$ for some $r \geqslant 1$. Therefore, $d = p^r d_0$ for some $r \geqslant 0$. □

Theorem 1. $C_d(n)$ admits a Hopf algebra structure if and only if there exist a d_0-th primitive root of unity $q \in k$ with $d_0 | n$ and a natural number $r \geqslant 0$ such that $d = p^r d_0$.

Proof. "If Part:" By Proposition 1, $\binom{d}{n}_q = 0$ for all $0 < n < d$. Now Example 1 implies the sufficiency.

"Only if Part:" If there is a Hopf structure on $C_d(n)$, then Lemma 3 implies that there is an n-th root of unity $q \in k$ such that

$$p_i^l \cdot p_j^m \equiv q^{jl} \binom{m+l}{l}_q p_{i+j}^{l+m} \pmod{C_{l+m}(n)}.$$

There is no harm to assume that q is a d_0-th primitive root of unity. Since the length of all paths in $C_d(n)$ is strictly less than d, $\binom{m+l}{l}_q = 0$ for all $0 < l, m < d$ and $m + l \geqslant d$. In particular, $\binom{d}{n}_q = 0$ for all $0 < n < d$. Thus by Proposition 1, $d = p^r d_0$ for some $r \geqslant 0$. □

Theorem 2. Any graded Hopf structure (with length grading) on $C_d(n)$ is isomorphic to some $C(d_0, t, n, q)$, where $C(d_0, t, n, q)$ is given as in Example 1.

Proof. By Lemma 3 and the proof of Theorem 1, we see that any graded Hopf structure (with the length grading) is isomorphic to $C(d_0, t, n, q)$ for some d_0-th primitive root of unity q with $d_0 | n$ and $d = p^t d_0$. □

The following example will show that there exist non-graded Hopf structures on $C_d(n)$. But, we cannot obtain a complete classification in this case.

Example 2. We give a non-graded Hopf structure on $C_{pd_0}(n)$. Let $q \in k$ be a d_0-th primitive root of unity with $d_0 | n$. As usual, we denote by p_i^l the path in Z_n of length l staring at e_i. For $s_1 d_0 + r_1 < pd_0$ and $s_2 d_0 + r_2 < pd_0$, we define the product as follows:

if $r_1 + r_2 \geqslant d_0$, then $p_i^{s_1 d_0 + r_1} p_j^{s_2 d_0 + r_2} = 0$;

if $r_1 + r_2 < d_0$ and $(s_1 + s_2) d_0 + r_1 + r_2 < pd_0$, then

$$p_i^{s_1 d_0 + r_1} p_j^{s_2 d_0 + r_2} = q^{r_1 j} \begin{pmatrix} (s_1 + s_2) d_0 + r_1 + r_2 \\ s_1 d_0 + r_1 \end{pmatrix}_q p_{i+j}^{(s_1+s_2)d_0 + r_1 + r_2};$$

if $r_1 + r_2 < d_0$ and $(s_1 + s_2) d_0 + r_1 + r_2 \geqslant pd_0$, then

$$\begin{aligned} p_i^{s_1 d_0 + r_1} p_j^{s_2 d_0 + r_2} = & q^{r_1 j} \frac{((d_0)!_q)^p ((s_1 + s_2) d_0 + r_1 + r_2)!_q}{(s_1 d_0 + r_1)!_q (s_2 d_0 + r_2)!_q} \cdot \\ & (p_{i+j}^{(s_1+s_2-p)d_0 + r_1 + r_2} - p_{i+j+pd_0}^{(s_1+s_2-p)d_0 + r_1 + r_2}). \end{aligned}$$

The antipode was given by

$$S(p_i^l) := (-1)^l q^{-\frac{l(l+1)}{2} - il} p_{n-l-i}^l$$

for $l \leqslant pd_0$. This is indeed a Hopf algebra with an identity element $p_0^0 = e_0$ and note that it is not graded with respect to the length grading. We can see that, as an algebra, it is generated by p_1^0, p_0^1 and $p_0^{d_0}$. An advantage of this construction is that we have a natural basis. We can also get this Hopf algebra through generators and relations.

Let n, d_0, p, q be as the above. We define $A(n, d_0, p, q)$ as follows. As an algebra, it is generated by g, x, y with relations

$$g^n = 1, \quad x^{d_0} = 0, \quad y^p = 1 - g^{pd_0}, \quad xg = qgx, \quad yg = gy, \quad yx = xy.$$

Its comultiplication Δ, counit ε and the antipode S are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x;$$

$$\Delta(y) = y \otimes 1 + g^{d_0} \otimes y + \Sigma_{i=1}^{d_0-1} \frac{1}{(d_0-i)!_q (i)!_q} g^i x^{d_0-i} \otimes x^i;$$

$$\varepsilon(g) = 1, \ \ \varepsilon(x) = \varepsilon(y) = 0;$$

$$S(g) = g^{n-1}, \ \ S(x) = -g^{n-1}x, \ \ S(y) = -g^{n-d_0}y.$$

Through a tedious but straightforward computation, we can prove $A(n, d_0, p, q)$ is indeed a Hopf algebra. We can also see that $A(n, d_0, p, q) \cong C_{pd_0}(n)$ as Hopf algebras by $g \mapsto p_1^0$, $x \mapsto p_0^1$ and $y \mapsto p_0^{d_0}$.

Let $q \in k$ be a d_0-th primitive root of unity with $d_0|n$, then Theorem 1 implies that there is a filtration of sub-Hopf algebras of $kZ_n(q)$:

$$C(d_0, n, q) \subset C(pd_0, n, q) \subset C(p^2 d_0, n, q) \subset \cdots \subset C(p^t d_0, n, q) \subset \cdots. \qquad (*)$$

Noth that, if $d_0 = 1$, $C_1(n)$ is indeed not a monomial coalgebra since it does not contain any arrow. But it is a Hopf algebra and clearly isomorphic to a group algebra.

If $d_0 \geqslant 2$, then $C_{d_0}(n)$ contains all vertices and arrows. By Lemma 1, all group-like and primitive elements of $kZ_n(q)$ lie in $C_{d_0}(n)$. Thus any path β whose length is no less than d_0 cannot be generated by the group-like and primitive elements since $\beta \notin C_{d_0}(n)$. Therefore, if $d_0 \geqslant 2$, then for any $t \geqslant 1$, $C(p^t d_0, n, q)$ cannot be generated by the group-like and primitive elements as Hopf algebras. This supplies many counter-examples for the following conjecture, which was posed by Andruskiewitsch and Schneider in ref. [7], when the characteristic of the base field is positive.

Andruskiewitsch-Schneider Conjecture: Let H be a finite-dimensional pointed Hopf algebra over an algebraically closed field of characteristic zero, then it is generated by the group-like and primitive elements.

Let n be a positive integer and let q be a d_0-th primitive root of unity with $d_0|n$. When the characteristic of k is zero, above Hopf algebras filtration $(*)$ will not happen. In fact, in ref. [1], the authors have shown (see the proof of Theorem 3.1 of ref. [1]) that in the filtration $(*)$, only when $d = d_0$, $C_d(n)$ is closed under a multiplication, i.e. $C(d_0, n, q)$ is the unique finite dimensional sub-Hopf algebra of $kZ_n(q)$ in the filtration. Thus we cannot deny the above conjecture since $C_{d_0}(n)$ is indeed generated by the group-like and primitive elements (see Theorem 3.6 in ref. [1]).

4 On monomial Hopf algebras

The main aim of this section is to discuss the structures of monomial Hopf algebras. Recall that a Hopf algebra is monomial if it is monomial as a coalgebra. We firstly prove a result which is similar to Theorem 5.1 in ref. [1].

Lemma 6. Let C be a monomial coalgebra. Then C admits a Hopf algebra structure if and only if as a coalgebra, $C \cong k \oplus \cdots \oplus k$ or

$$C \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

for some $d = p^r d_0 \geqslant 2$ with $d_0 | n$ and there exists a d_0-th primitive root of unity $q \in k$.

The proof of this lemma is also similar to that of Theorem 5.1 in ref. [1]. For convenience to the readers, we write it down.

Proof. "If Part:" By assumption, we have $C = C_1 \oplus \cdots \oplus C_l$ as a coalgebra, where each $C_i \cong C_1$ as coalgebras for $1 \leqslant i \leqslant l$ and C_1 admits a Hopf structure H_1 by Theorem 1. Then $H_1 \otimes kG$ gives a Hopf structure on C, where G is any group of order l. This gives the sufficiency.

"Only if Part:" Let C be a monomial coalgebra admitting a Hopf structure. Since a finite-dimensional Hopf algebra is coFrobenius, it follows from Lemma 2 that as a coalgebra C has the form $C = C_1 \oplus \cdots \oplus C_l$ with each C_i indecomposable as a coalgebra, and $C_i = k$ or $C_i = C_{d_i}(n_i)$ for some n_i and $d_i \geqslant 2$.

We claim that if there exists some $C_i = k$, then $C_j = k$ for all j. In fact, otherwise, let $C_j = C_d(n)$ for some j. Let α be an arrow in C_j from x to y, then α gives a non-trivial x, y-primitive element in C. Let h be the unique group-like element in $C_i = k$. Since the set $G(C)$ of the group-like elements of C forms a group, it follows that $hx^{-1}\alpha$ is a non-trivial h, yhx^{-1}-primitive element in C. But according to the coalgebra decomposition $C = C_1 \oplus \cdots \oplus C_l$ and $C_i = k \cdot h$, C contains no non-trivial h, gy-primitive elements. A contradiction.

Thus, by the above claim, if $C \neq k \oplus \cdots \oplus k$, then

$$C = C_{d_1}(n_1) \oplus \cdots \oplus C_{d_l}(n_l)$$

as coalgebras, with each $d_i \geqslant 2$. Assume that the identity element 1 of $G(C)$ is contained in the component $C_1 = C_{d_1}(n_1)$. It follows from a theorem of Montgomery (see Theorem 3.2 in ref. [4]) that C_1 is a sub-Hopf algebra of C, and that as coalgebras

$$g_i^{-1} C_{d_i}(n_i) = C_{d_i}(n_i) g_i^{-1} = C_{d_1}(n_1)$$

for any $g_i \in G(C_{d_i}(n_i))$ and for each i. By comparing the numbers of the group-like elements in $C_{d_i}(n_i)$ and in $C_{d_1}(n_1)$, we have $n_i = n_1 = n$ for each i while by comparing the k-dimensions, we see that $d_i = d_1 = d$ for each i. Now, since $C_1 = C_d(n)$ is a Hopf algebra, by Theorem 1, there exists a d_0-th primitive root of unity $q \in k$ with $d_0 | n$ and $r \geqslant 0$ such that $d = p^r d_0$. □

Theorem 3. Let H be a non-semisimple monomial Hopf algebra over k. Then there exists a d_0-th primitive root of unity $q \in k$ with $d_0 | n$, $r \geqslant 0$ and $d = p^r d_0 \geqslant 2$ such that

$$H \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

as coalgebras; and

$$H \cong C(d, n, q) \#_\sigma k(G/N)$$

as Hopf algebras, where $G = G(H)$ and $N = G(C(d, n, q))$.

Proof. By Theorem 3.2 in ref. [4] and Lemma 6 above, we can get this conclusion directly. $\square$

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